Section 3.6 *A Summary of Curve Sketching*

Analyzing the Graph of a Function

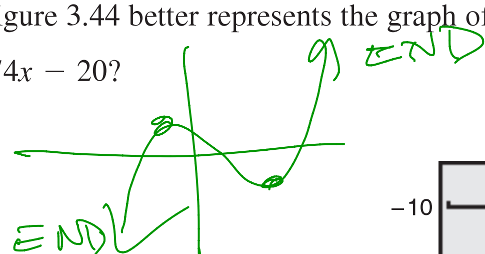
It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

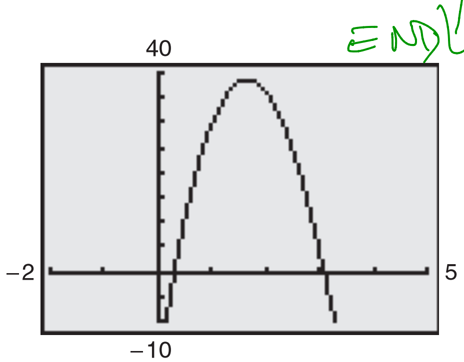
So far, you have studied several concepts that are useful in analyzing the graph of a function.

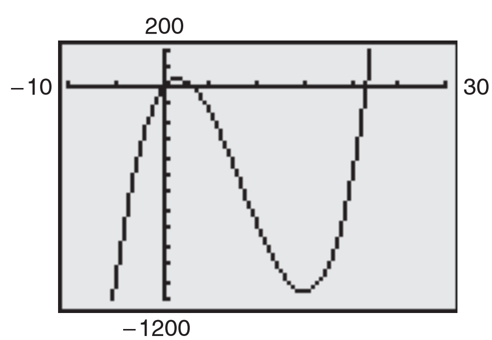
- x-intercepts and y-intercepts (Section P.1) Pre
- Symmetry (Section P.1) Pre
- Domain and range (Section P.3) Pre
- Continuity (Section 1.4) Pre
- Vertical asymptotes (Section 1.5) Pre
- Differentiability (Section 2.1) Calc
- Relative extrema (Section 3.1) Calc
- Concavity (Section 3.4) Calc
- Points of inflection (Section 3.4) Calc
- Horizontal asymptotes (Section 3.5) Pre
- Infinite limits at infinity (Section 3.5) Pre/Calc

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing windows in Figure 3.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

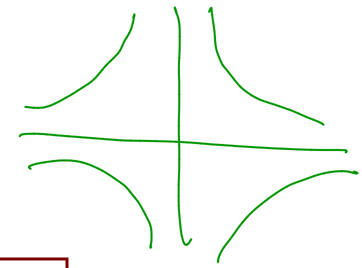






Different viewing windows for the graph of $f(x) = x^3 - 25x^2 + 74x - 20$
Figure 3.44

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function.



Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

NOTE In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$. ■

Ex.1 Analyze and sketch the graph of $f(x) = \dfrac{2(x^2-9)}{x^2-4}$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

1st derivative: $f'(x) = \dfrac{20x}{(x^2-4)^2}$

2nd derivative: $f''(x) = \dfrac{-60x^2-80}{(x^2-4)^3}$

***x*-intercepts:** $(3,0)$ & $(-3,0)$

***y*-intercepts:** $(0, \frac{9}{2})$

vertical asymptotes: $x=2$ & $x=-2$

horizontal asymptotes: $y=2$

critical numbers: at $x=0$

possible points of inflection: None

$$f'(x) = \frac{(x^2-4)(4x) - 2(x^2-9)(2x)}{(x^2-4)^2}$$

$$f'(x) = \frac{4x^3-16x-4x^3+36x}{(x^2-4)^2}$$

$$f'(x) = \frac{20x}{(x^2-4)^2}$$

chain rule

$$f''(x) = \frac{[(x^2-4)^2]20 - 20x[2(x^2-4)^1\, 2x]}{[(x^2-4)^2]^2}$$

$$f''(x) = \frac{(x^2-4)^1[20(x^2-4)^1 - 80x^2]}{(x^2-4)^4}$$

$$f''(x) = \frac{20x^2-80-80x^2}{(x^2-4)^3}$$

$$f''(x) = \frac{-60x^2-80}{(x^2-4)^3}$$

x-intercepts

$f(x)=0$

$$0 = \frac{2(x^2-9)}{x^2-4}$$

$0 = (x^2-9)$

$0 = (x+3)(x-3)$

$x=-3$, or $x=3$

y-intercept

$x=0$

$y=f(0)$

$$y = \frac{2(0^2-9)}{0^2-4}$$

$$y = \frac{2\cdot(-9)}{-4}$$

$$y = \frac{9}{2} = 4.5$$

VA.

$x^2-4=0$

$(x-2)(x+2)=0$

$x-2=0$, or $x+2=0$

$\underline{x=2}$ $\underline{x=-2}$

HA.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2(x^2-9)}{x^2-4}$$

right

$\underline{y=2}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2(x^2-9)}{x^2-4}$$

left

$y=2$

critical numbers:

(A) $f'(x)=0$ or (B) $f'(x)$ is undefined

$$\frac{20x}{(x^2-4)^2}=0$$

$20x=0$

$x=0$

$x^2-4=0$

$x=\pm 2$

not on $y=f(x)$

Possible Points of inflection:

(A) $f''(x) = 0$ or (B) $f''(x)$ is undefined

$x^2 - 4 = 0$

$x = \pm 2$

not on

$y = f(x)$

$$\frac{-60x^2 - 80}{(x^2-4)^3} = 0$$

$-60x^2 - 80 = 0$

$-60x^2 = 80$

$x^2 = -\frac{80}{60}$

Never

Ex.1 cont'd

$f'(-3) = \dfrac{20(-3)}{((-3)^2-4)^2}$

$f'(-3) = \dfrac{-60}{+} < 0$

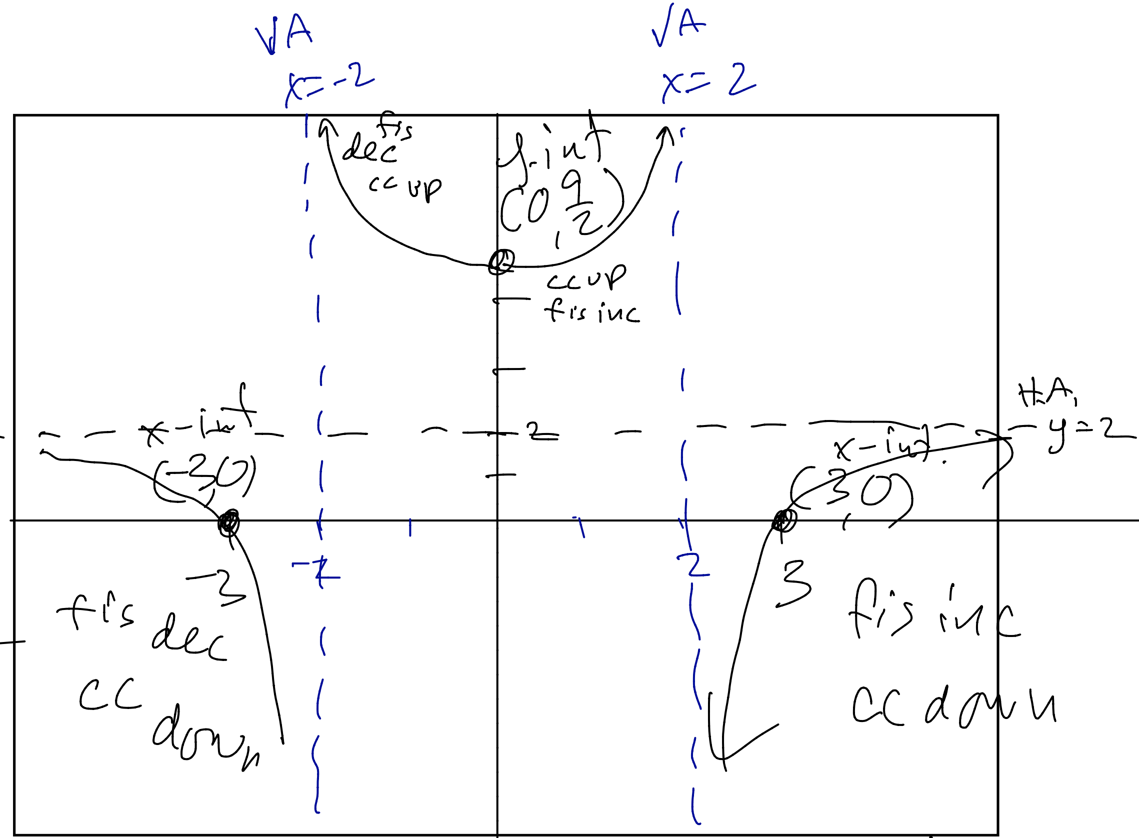
$f'(-1) = \dfrac{20(-1)}{((-1)^2-4)^2}$

$f'(-1) = \dfrac{-20}{+} < 0$

$f'(1) = \dfrac{20(1)}{((1)^2-4)^2}$

$f'(1) = \dfrac{20}{+} > 0$

$f'(3) = \dfrac{20(3)}{+} > 0$



$f''(-3) = \dfrac{-60(-3)^2-80}{((-3)^2-4)^3}$

$f''(-3) = \dfrac{-60\cdot 9-80}{(9-4)^3}$

$f''(-3) = \dfrac{-540-80}{5^3} < 0$

$f''(-1) = \dfrac{-60(-1)^2-80}{((-1)^2-4)^3}$

$f''(-1) = \dfrac{-60-80}{(1-4)^3}$

$f''(-1) = \dfrac{-140}{(-3)^3} > 0$

$f''(3) = \dfrac{-60(3)^2-80}{((3)^2-4)^3}$

$f''(3) = \dfrac{-60\cdot 9-80}{(9-4)^3}$

$f''(3) = \dfrac{-540-80}{5^3}$

$f''(3) = \dfrac{-620}{125} < 0$

L → R

Interval/ Point	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f(x)$	$f(-3) = 0$	$f(0) = \frac{9}{2}$		$f(3) = 0$
$f'(x)$	$f'(-3) < 0$ $-$	$f'(-1) < 0$ $-$	$f'(1) > 0$ $+$	$f'(3) > 0$ $+$
$f''(x)$	$f''(-3) < 0$ $-$	$f''(-1) > 0$ $+$	$+$	$f''(3) < 0$ $-$
Character -istic(s) of graph	f is dec f is ccdown	f is dec f is ccup	f is inc f is ccup	f is inc f is concave down

Ex.2 Analyze and sketch the graph of $f(x)=\dfrac{x^2-2x+4}{x-2}$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

1st derivative: $f'(x)=\dfrac{x^2-4x}{(x-2)^2}=\dfrac{x(x-4)}{(x-2)^2}$

2nd derivative: $f''(x)=\dfrac{8}{(x-2)^3}$

***x*-intercepts**: none

***y*-intercepts**: $(0,-2)$

vertical asymptotes: $x=2$

horizontal asymptotes:

critical numbers:

possible points of inflection:

$$f'(x)=\frac{(x-2)(2x-2)-(x^2-2x+4)(1)}{(x-2)^2}$$

$$f'(x)=\frac{2x^2-2x-4x+4-x^2+2x-4}{(x-2)^2}$$

$$f'(x)=\frac{x^2-4x}{(x-2)^2}$$

$$f''(x)=\frac{(x-2)^2\cdot(2x-4)-(x^2-4x)[2(x-2)^1\cdot 1]}{[(x-2)^2]^2}$$

$$f''(x)=\frac{2(x-2)[(x-2)^2-(x^2-4x)]}{(x-2)^4}$$

$$f''(x)=\frac{2\cdot[x^2-4x+4-x^2+4x]}{(x-2)^3}$$

$$f''(x)=\frac{8}{(x-2)^3}$$

x-intercepts $y=0$

$f(x)=0$

$\frac{x^2-2x+4}{x-2}=0$

$x^2-2x+4=0$

$x^2-2x=-4$

$x^2-2x+\underline{1}=-4+\underline{1}$

$\left(\frac{-2}{2}\right)^2=(-1)^2=1$

$(x-1)(x-1)=-3$

$(x-1)^2=-3$

False

none

y-intercepts $x=0$

$y=f(0)$

$y=\frac{(0)^2-2(0)+4}{(0)-2}$

$y=\frac{4}{-2}$

$y=-2$

$(0,-2)$

V.A.

$f(x)=\frac{x^2-2x+4}{x-2}$

$x-2=0$

$x=2$

Horizontal Asymptotes

"left"

$\lim_{x\to-\infty} f(x)$

$=\lim_{x\to-\infty}\frac{x^2-2x+4}{x-2}$

$=\lim_{x\to-\infty}\left[\frac{\frac{x^2-2x+4}{1}}{\frac{x-2}{1}}\right]\left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]$

$=\lim_{x\to-\infty}\frac{x-2+\frac{4}{x}}{1-\frac{2}{x}}$

$=\frac{-\infty-2+0}{1-0}$

$=-\infty$

unbounded down

"Right"

$\lim_{x\to\infty} f(x)$

$=\lim_{x\to\infty}\frac{x^2-2x+4}{x-2}$

$=\lim_{x\to\infty}\left[\frac{\frac{x^2-2x+4}{1}}{\frac{x-2}{1}}\right]\left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]=\lim_{x\to\infty}\frac{x-2+\frac{4}{x}}{1-\frac{2}{x}}$

$=\frac{\infty-2+0}{1-0}=\infty$

unbounded up

Ex.2 cont'd

Interval/ Point	
$f(x)$	
$f'(x)$	
$f''(x)$	
Character-istic(s) of graph	

$f(x) = \dfrac{x}{(x^2+2)^{1/2}}$

Ex.3 Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2+2}}$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

$f'(x) = \dfrac{(x^2+2)^{1/2}\cdot(1) - x\left[\frac{1}{2}(x^2+2)^{-1/2}\,2x\right]}{(\sqrt{x^2+2}\;)^2}$

$f'(x) = \dfrac{(x^2+2)^{-1/2}\cdot\left[(x^2+2)^{2/2} - x^2\right]}{(x+2)^{2/2}}$

$f'(x) = \dfrac{2}{(x+2)^{3/2}}$ or $f'(x) = 2(x^2+2)^{-3/2}$

$f''(x) = 2\cdot\left[\frac{-3}{2}(x^2+2)^{-5/2}\cdot 2x\right]$

$f''(x) = -6x(x^2+2)^{-5/2}$ or $f''(x) = \dfrac{-6x}{(x^2+2)^{5/2}}$

1st derivative: $f'(x) = \dfrac{2}{(x^2+2)^{3/2}}$ or $f'(x) = 2(x^2+2)^{-3/2}$

2nd derivative: $f''(x) = \dfrac{-6x}{(x^2+2)^{5/2}}$

***x*-intercepts**: ______

***y*-intercepts**: ______

vertical asymptotes: ______

horizontal asymptotes: ______

critical numbers: ______

possible
points of inflection: ______

Ex.3 cont'd

Interval/ Point	
$f(x)$	
$f'(x)$	
$f''(x)$	
Character-istic(s) of graph	

Ex.4 Analyze and sketch the graph of $f(x)=2x^{\frac{5}{3}}-5x^{\frac{4}{3}}$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

$f'(x) = 2\cdot\frac{5}{3}x^{2/3} - 5\cdot\frac{4}{3}x^{1/3} = \frac{10}{3}x^{2/3} - \frac{20}{3}x^{1/3}$

$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3}-2)$

1st derivative: $f'(x)=\frac{10}{3}x^{1/3}(x^{1/3}-2)$

$f''(x) = \frac{10}{3}\cdot\frac{2}{3}x^{-1/3} - \frac{20}{3}\cdot\frac{1}{3}x^{-2/3}$

$f''(x) = \frac{20}{9}x^{-2/3}(x^{1/3}-1)$

$f''(x) = \frac{20}{9}x^{-2/3}(x^{1/3}-1)$

or

$f''(x) = \frac{20(x^{1/3}-1)}{9x^{2/3}}$

2nd derivative: $f''(x) = \frac{20(x^{1/3}-1)}{9x^{2/3}}$

***x*-intercepts**: ______

***y*-intercepts**: ______

vertical asymptotes: ______

horizontal asymptotes: ______

critical numbers: ______

possible points of inflection: ______

Ex.4 cont'd

Interval/ Point	
$f(x)$	
$f'(x)$	
$f''(x)$	
Character-istic(s) of graph	

Ex.5 Analyze and sketch the graph of $f(x)=x^4-12x^3+48x^2-64x$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

$f'(x)=4x^3-12\cdot 3x^2+48\cdot 2x^1-64$

$f'(x)=4x^3-36x^2+96x-64$

$f''(x)=4\cdot 3x^2-36\cdot 2x^1+96$

$f''(x)=12x^2-72x+96$

1st derivative: $f'(x)=4x^3-36x^2+96x-64$

2nd derivative: $f''(x)=12x^2-72x+96$

***x*-intercepts**: ______

***y*-intercepts**: ______

vertical asymptotes: ______

horizontal asymptotes: ______

critical numbers: ______

possible points of inflection: ______

Ex.5 cont'd

Interval/ Point	
$f(x)$	
$f'(x)$	
$f''(x)$	
Characteristic(s) of graph	

Ex.6 Analyze and sketch the graph of $f(x)=\dfrac{\cos(x)}{1+\sin(x)}$. Fill in the analysis chart on the next page. When partitioning the x-axis in the chart, include and test all critical numbers and possible points of inflection. Use ***calculus techniques and algebra*** to show the following:

1st derivative: $f'(x) = \dfrac{-1}{\sin(x)+1}$

2nd derivative: $f''(x) = \dfrac{\cos(x)}{[1+\sin(x)]^2}$

***x*-intercepts**:

***y*-intercepts**:

vertical asymptotes:

horizontal asymptotes:

critical numbers:

possible points of inflection:

$$f'(x) = \frac{[1+\sin(x)][-\sin(x)] - [\cos(x)][\cos(x)]}{[1+\sin(x)]^2}$$

$$f'(x) = \frac{-\sin(x) - \sin^2(x) - \cos^2(x)}{[1+\sin(x)]^2}$$

$$f'(x) = \frac{-\sin(x) - [\sin^2(x)+\cos^2(x)]}{[1+\sin(x)]^2}$$

$$f'(x) = \frac{-\sin(x) - 1}{[1+\sin(x)]^2}$$

$$f'(x) = \frac{-1\cdot[\sin(x)+1]}{[1+\sin(x)]^2}$$

$$f'(x) = \frac{-1}{1+\sin(x)} \quad \text{or} \quad f'(x) = -(1+\sin(x))^{-1}$$

$$f''(x) = -1\cdot[-1(1+\sin(x))^{-2}]\cdot\cos(x)$$

$$f''(x) = \frac{\cos(x)}{[1+\sin(x)]^2}$$

Ex.6 cont'd

Interval/ Point	
$f(x)$	
$f'(x)$	
$f''(x)$	
Character -istic(s) of graph	